Discrete Prolate Spheroidal Sequence Based Filter Banks for the Analysis of Nonstationary Signals

Azime Can-Cimino
Swanson School of Engineering
University of Pittsburgh
Pittsburgh, PA 15261
Email: azime.cancimino@pitt.edu

Luis F. Chaparro
Swanson School of Engineering
University of Pittsburgh
Pittsburgh, PA 15261
Email: chaparro@pitt.edu

Ervin Sejdić
Swanson School of Engineering
University of Pittsburgh
Pittsburgh, PA 15261
Email: esejdic@ieee.org

***Abstract*—Filter banks are often used in the analysis and the synthesis of signals. By using cascade or parallel connected filters, multi-resolution analysis of a signal in different sub-bands is possible. A method based on the filter banks using discrete prolate spheroidal sequences (DPSS) is proposed in this paper. DPSS are solutions to an energy maximization problem in a limited bandwidth context; their time-frequency concentration aspect is exploited in the proposed design. A filter bank is derived using DPSS sequences and their modulated counter parts and is utilized in the analysis of non-stationary signals. The designed filter-bank achieves a concise representation of a non-stationary signal by maximizing the spectral energy in each sub-band.**

I. Introduction

Non-uniform samples appear in many signal analysis applications as a result of signal-dependent sampling algorithms [1], [2], [3], [4], [5]. The density of the samples drawn by any of the signal-dependent sampling algorithms increases with the signal's activity in time. Although these sampling approaches have been shown to be successful in capturing the time-frequency behavior of a signal, the distribution of the acquired samples challenges the reconstruction process. In regions where the signal activity is high, enough samples are returned to retrieve information perfectly. Conversely, the idle parts of the signal may be underrepresented by the generated samples, and hence their recovery is not as trivial. This phenomenon encountered in reconstruction from non-uniform samples displays a well recognized ill-posed problem. Synchronous sampling at a Nyquist rate can diminish this challenge, but the number of the samples returned for a non-stationary signal is highly redundant, especially when the signal's spectrum expands to a wide range of frequencies [6]. Many regularization-based practical solutions can be found for the recovery of non-stationary signals from non-uniform samples in the literature [7], [8].

A subset of the non-stationary signals, namely sparse signals, are not subject to above-mentioned reconstruction challenge. This is mainly due to the the fact that the idle parts of the signal generally do not bear much, if any information. For this case, a trivial assumption of no change in between two consecutive samples of an idle part is reasonable. Also, periodic samples are returned as a result of a stationary behavior, e.g., an oscillation at one frequency. These types of sparse signals widely appear in sensing based applications, particularly in bio-sensing and sensor networks. An asynchronous sampler can exploit signal dependency and yet not get affected from the aforementioned difficulty in the recovery step. The sampling methods for the representation of sparse signals, as well as efficient reconstruction techniques are well studied [9]. They fail to scale to a wider class of non-stationary signals, and stable reconstruction remains a challenge.

An unequivocal way to approach the aforementioned problem is to consider the sparsification of non-stationary signals, which allows one to take advantage of the methods used in the reconstruction of sparse signals [10]. Biomedical signals are the main scope of interest in this work. One common aspect of biomedical signals is that information of interest is often a combination of features that are well localized temporarily or spatially (e.g spikes and transients in EEG) and others that are more dispersed (small oscillations, bursts). These characteristics call for the use of analysis methods that are sufficiently versatile to handle events that can be at opposite extremes in terms of their time-frequency localizations.

In this paper, we propose a scheme that adapts to different components of a non-stationary signal; enabling us to look at brief high frequency components as well as long-lived low frequency components. We derive a scale-based filtering scheme using DPSS filters which returns a small number of scale parameters and simplifies the reconstruction step. The following section introduces the analysis function used in the filters. Section III contains the details of the proposed method, and we then present a simulation case for an EKG signal using the proposed method in Section IV. The conclusions are drawn in the last section of this paper.

II. Analysis Functions

The proposed analysis technique adopts discrete prolate spheroidal sequences (DPSS) that are the discrete counter part of prolate spheroidal wave functions, also known as Slepian functions [11]. DPSS are parametrized by a time-bandwidth product NW, where N is the length of the sequence and W is the bandwidth in radians. These sequences are defined as the solution to the following eigenvalue problem [11]:

$$\lambda_k \phi_k(m) = \sum_{n=0}^{N-1} \frac{sin(2nW(n-m))}{n(n-m)} \phi_k(n)$$

where $0 \leq n, m \leq N-1$, and $0 \leq W \leq 1/2$. The N real eigenfunctions $\phi_k(n)$ are the DPSS, and the corresponding eigenvalues relate to their energy concentration. The DPSS are doubly orthogonal, that is, they are orthogonal on the infinite set $\{-\infty, ..., \infty\}$ and orthonormal on the finite set $\{0, 1, ..., N-1\}$, that is,

$$\sum_{-\infty}^{\infty} \phi_i(n, N, W)\phi_j(n, N, W) = \lambda_i \delta_{ij}$$
$$\sum_{n=0}^{N-1} \phi_i(n, N, W)\phi_j(n, N, W) = \delta_{ij}$$

where $i, j = 0, 1, ..., N-1$. The sequences also obey symmetry laws

$$\phi_k(n, N, W) = (-1)^k \phi_k(N-1-n, N, W)$$
$$\phi_k(n, N, W) = (-1)^k \phi_{N-1-k}(N-1-n, N, 1/2-W)$$

where $n = 0, \pm1, \pm2, ...$ and $k = 0, 1, ..., N-1$. The first eigenfunction corresponding to the maximum eigenvalue, $\phi_1(n)$, is the most concentrated in the frequency band W. This sequence is used as a basis through the rest of this work, and we disregard the remaining DPSS sequences in our analysis.

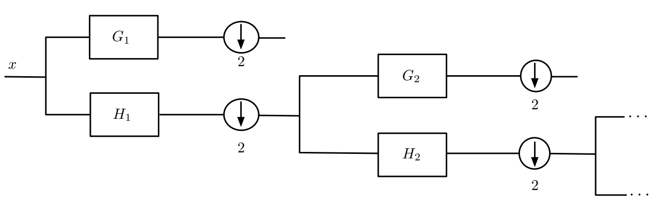

Fig. 1. Bank of filters for discrete wavelet transform with a scale factor of 2. The low-pass filters H_j, and high pass filters G_j where $j = 1, 2, \ldots, L$, and L total number of scales. $\downarrow 2$ represents down-sampling by 2.

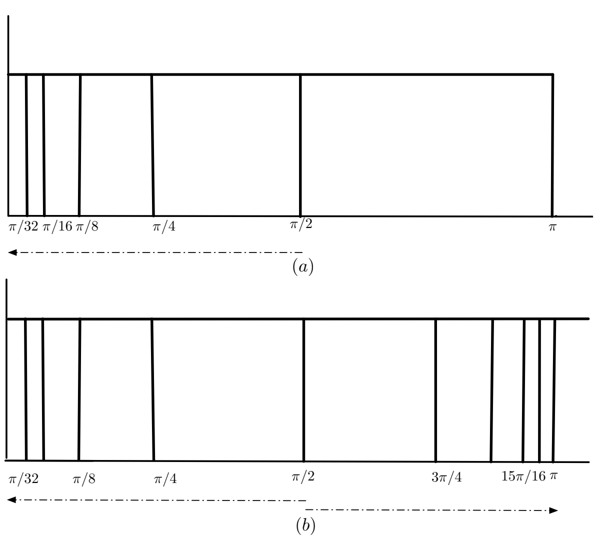

Fig. 2. Division of frequency spectrum: (a) by low-pass expanding filter bank, (b) by low- and high- frequency expanding filter-bank

DPSS sequences, being solutions to this energy concentration problem, are maximally concentrated in a time interval, i.e. $[-T/2, T/2]$ and they are also band-limited [14], [15], [16]. In other words, they have high energy concentration both in time and in frequency. This time-frequency concentration feature gave rise to their use in signal compression, representation, and reconstruction [12], [13], [17]. The same feature makes these sequences optimum basis candidates for discrete wavelet analysis. The smoothness and the time support of the basis are critical features in wavelet analysis; waveforms that are smooth and have finite support are most desired [18]. The smoothness of the basis is particularly important in the scope of this work. Non-uniform samples that appear as a result of signal dependent sampling are denser in the high-activity regions and scarce in the slow changing regions of the signal. Hence, a smooth basis can approximate low-density parts very well.

III. Method: Filter Banks

The proposed method for the analysis of non-stationary signals is based on filter banks that are formed using DPSS sequences. Expansion of signals in terms of wavelets and perfect reconstruction filter banks are proven to be equivalent forms of signal representation; we have adapted the latter one because it yields implementable systems [19]. Filter banks are generally formed using a set of high and low-pass finite impulse response (FIR) filters connected in cascade or parallel. The configurations enable the analysis of a signal on sub-bands that are formed by band-selective filters used in the design. A common arrangement of a filter bank used in a wavelet transform is depicted in Fig.1. The main feature of this configuration is that it focuses gradually on the low-end part of the frequency spectrum, see Fig. 2(a). One can also design a configuration that zooms-in more towards high-frequencies by expanding the branch on the high pass filters; or alternatively one can expand the filter-bank on both ends to obtain a splitting depicted in Fig. 2(b). Depending on the application, many other configurations of the filters are possible that lead to different splitting patterns in frequency. Note that the spectrum we depicted in Fig. 2(b) has a dyadic splitting on the frequency axis that can be achieved by letting the length of the cascaded filters increase by two at every cascaded branch, i.e. the first set of filters have length 2^n and the sequential ones will have length 2^{n+1}.

The filter denoted by H_j, named as a *s*caling filter, is an FIR low-pass filter, $j = 1, 2, \ldots, L$ where L is the total number of scales used in the filter bank design. In this work, these filters' coefficients are obtained from a DPSS sequence of the length N_j^h and the frequency support of the sequence is W_j^h. On the other hand, the filter denoted by G_j exhibits a high-pass behavior, which also has a finite-length, and is named as a *w*avelet filter. Similar to the scaling filter, the wavelet filter G_j is derived from a DPSS sequence of a length N_j^g and a frequency support of W_j^g. In a typical wavelet filter bank arrangement, the frequency support of the first filters are designed to sum up to the maximum discrete frequency.

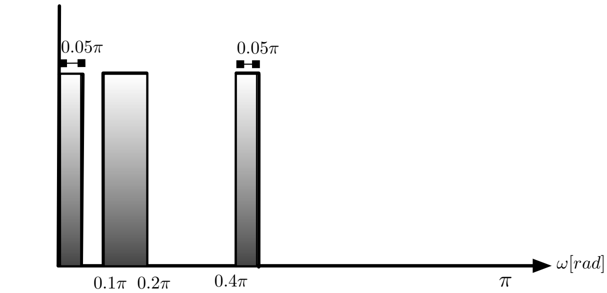

Fig. 3. Example signal spectrum

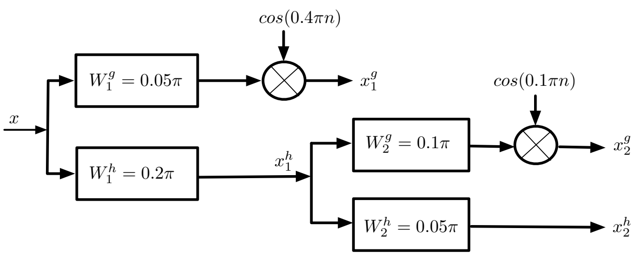

Fig. 4. Filter-bank designed for the example in figure 3

For example, if $W_1^h = 0.4\pi$ and we assume that the highest discrete frequency present is π radians at the Nyquist rate, the high pass filter should have a bandwidth of 0.6π, i.e. $W_1^g = 0.6\pi$ in order for both filters to sum up to the whole spectrum. This is desired if we want to capture the whole frequency content of the signal; otherwise any initial value for the filters can be used. Then, the preceding filters are designed accordingly, e.g. for dyadic splitting of the low frequencies $W_2^h = W_1^h/2 = 0.1\pi$, $W_3^h = 0.05\pi$, and so on. This way a low-frequency content of the signal can be fully represented/analyzed by the filter bank. Similarly, the same approach can be applied to the high-frequencies by expanding the bank on the high-pass end. By using a bank

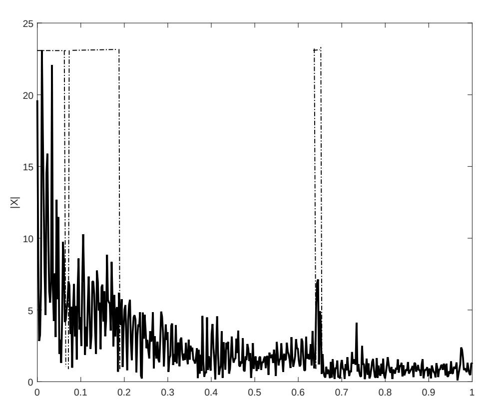

Fig. 5. Frequency spectrum of an EKG signal

of filters, a multi resolution analysis of the signals' is possible [20]. Conventionally, scales goes from coarser to finer, and as scale increases more details about the signal can be revealed. In this study, our interest is the multi-resolution spaces that are formed by the DPSS basis and their bi-orthogonal pairs [21]. We exploit the time-frequency concentration aspect of the DPSS sequences both in the analysis and synthesis filter banks. In order to explain the difference of our filter bank configuration and the utilization of DPSS sequences, consider the following toy example. Suppose we are given a signal with a frequency support shown in Fig. 3, and we want to analyze this signal as concisely as possible. With usage of the filter bank tool, this goal translates into "a minimum number of scales" that can be employed in the representation.

If a dyadic splitting is picked and it did not provide an optimum outcome in minimizing the number of scales, one can change the scaling-ratio to 3 or 4, etc. This will create a different step size in the frequency split, and maybe the regions of target (the frequency region where the signal lays) can be arrived faster (i.e. less number of scales). Although changing the scaling-ratio may provide an ad-hoc solution, the overall pattern in the frequency splitting remains the same even when a optimum-ratio is adopted. One still needs to start with a coarser scale to capture the coarse components of the given signal, and then proceed into finer scales to able to zoom in to the regions of details. This can be regarded as a constraint if the conciseness of the representation is the interest, i.e. decrease in the number of scales in the representation. Motivating from this constraint, we propose an alternative filter bank scheme that offers more flexibility in reaching the final frequency target in a small number of scales.

In order to decrease the number of scales in the representation while preserving its accuracy, the key is to design filters that could exactly match the frequency support of the analyzed signal. The assumption is that the frequency content of the signal is known a-priori. As highlighted in the previous section, the first DPSS sequence is optimum in maximizing the energy concentration to a limited bandwidth, yet with still a finite length in time. By adapting the frequency support of the sequences to the target frequency regions, a perfect filter bank can be derived for a given signal. In the case of our toy example, H_1's coefficients are picked using the first DPSS sequence that gives the maximum energy concentration with a frequency support of $W_1^h = 0.05\pi$, and filter G_1 is derived from a sequence with a bandwidth of $W_1^g = 0.2\pi$. The overall design for this particular signal is depicted in Fig. 4.

Note that, a filter bank that is formed by any good (smooth) wavelet basis can do the representation accurately by using finer scales. This is true for the proposed method as well; by increasing the number of filters and hence the scales, more and more detailed approximations are also possible. Though this creates an additional computation cost and higher complexity of the representation. The main advantage over a general filter bank approach is that filter banks that are formed by DPSS enable one to maximize the spectral energy at a given scale which provides a more concise representation of a non-stationary signal.

IV. Simulation

In this section, we simulate the proposed filter bank in the analysis of an EKG signal with a frequency spectrum depicted by Fig. 5. The dashed boxes in this figure are indicating the frequency ranges where most of the signal's energy is concentrated and the focus in the filter bank design is put on those regions for a concise representation. One could also design a filter bank to comprise the whole signal spectrum. The filters are designed to achieve high accuracy with little number of expansions, i.e. scales, for the given non-stationary signal. The overall design includes 2 scales and 4 filters; the output from each filter is depicted in Fig. 6.

The reconstructed signal as a sum of the three decomposed parts, $y_1^h + y_2^h + y_1^g$, is shown in the following Fig. 7. In here, y_1^h is the output from H_1, y_2^h is the output from H_2 and y_1^g is the output from G_1. Perfect reconstruction is also possible by using synthesis filters that are inverses of the DPSS analysis filters. The de-noising of the signal is achieved by ignoring the last component from the second high-pass filter G_2. The reconstructed signal, Fig. 7 bottom, is free of background noise as we disregard most of the noise, y_2^g in Fig. 6. The representation of the EKG signal considering only the frequency bands where most of the signal's energy is, enables a concise representation.

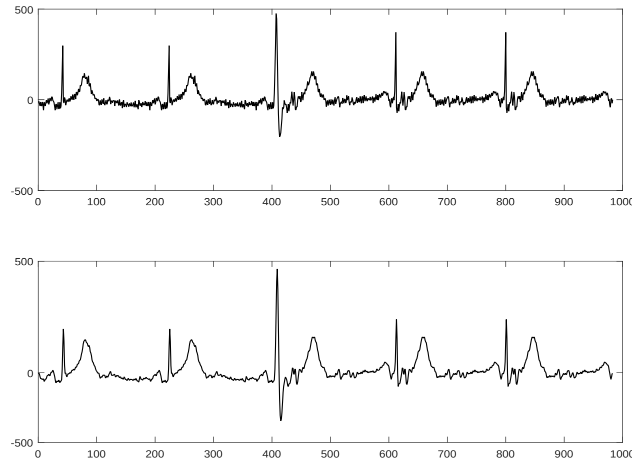

Fig. 7. Input signal top, and the reconstructed signal bottom

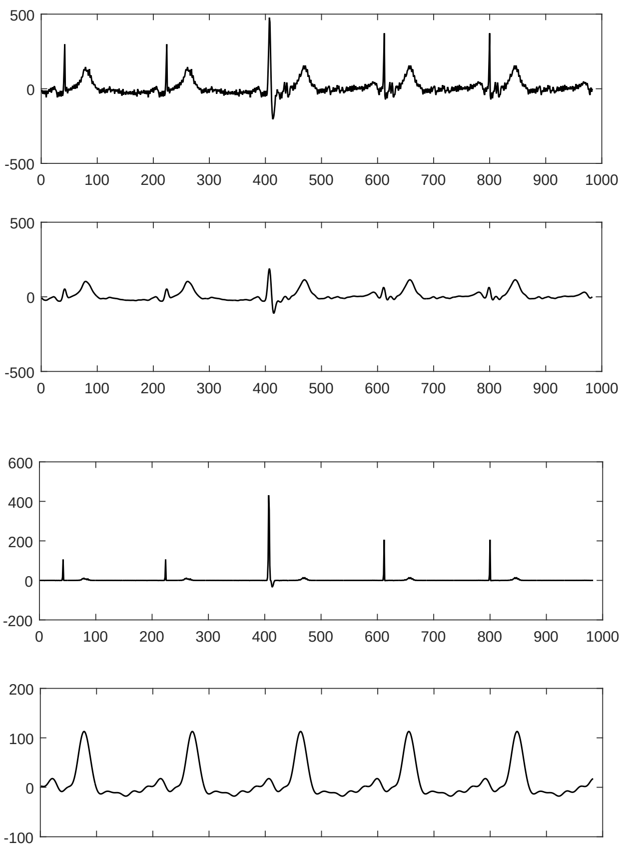

Fig. 6. Filters' outputs respectively from: (a) filter H_1, (b) filter G_1, (c) filter H_2, and (d) filter G_2

V. Conclusion

In this paper, we proposed a scheme for filter bank design that adapts to different components of a non-stationary signal; enabling one to look at brief, high frequency components as well as long-lived low frequency components. We derive a scale based filtering scheme using DPSS filters which returns a small number of scale parameters. By exploiting the time-frequency concentration of DPSS sequences, a concise representation is attained.

References

[1] Y. Tsividis, "Event-Driven Data Acquisition and Digital Signal Processing: a Tutorial", *IEEE Trans. on Circuits and Systems II: Express Briefs*, Vol. 57, pp. 577-581, Aug. 2010.

[2] Y. Tsividis, "Digital signal processing in continuous time: a possibility for avoiding aliasing and reducing quantization error," *IEEE Intl. Conf. on Acoust., Speech, and Sig. Proc. (ICASSP)*, vol. II, pp. 589-92, May 2004.

[3] E.V. Aksenov, Yu.M. Ljashenko, A.V. Plotnikov, D.A. Prilutskiy, S.V. Selishchev and E.V. Vetvetskiy, "Biomedical data acquisition systems based on sigma-delta analogue-to-digital converters," *Intl. Conf. of the IEEE Engineering in Medicine and Biology Society*, vol.4, pp. 3336-3337, 2001.

[4] A. Can and E. Sejdić and L. Chaparro, "An asynchronous scale decomposition for biomedical signals", *IEEE Sig. Proc. in Medicine and Biol. Symp*, pp. 1-6, 2011.

[5] A. Can and E. Sejdić and O. Alkishriwo and L. Chaparro, "Compressive asynchronous decomposition of heart sounds", *IEEE Statistical Signal Processing Workshop (SSP)*, pp. 736-739, Aug. 2012.

[6] Kinniment, D. and Yakovlev, A. and Gao, B., "Synchronous and asynchronous A-D conversion", *IEEE Transactions on Very Large Scale Integration (VLSI) Systems*, pp. 217-220, Vol.8, 2000.

[7] A. Lazar, and L. Toth, "Time encoding and perfect recovery of bandlimited signals," *IEEE Intel. Conf. on Acous., Speech, and Sig. Proc. (ICASSP)*, VI- 709-12, Apr. 2003.

[8] S. Senay, L.F. Chaparro, M. Sun and R.J. Sclabassi, "Adaptive Level-Crossing Sampling and Reconstruction," *18th European Signal Processing Conference*, Aug. 2010.

[9] A. Can, E. Sejdić and L. Chaparro, "Asynchronous sampling and reconstruction of sparse signals", *Proceedings of the 20th European Signal Processing Conference (EUSIPCO)*, pp. 854-858, Aug. 2012

[10] L. Chaparro, E. Sejdić, A. Can, O. Alkishriwo, S. Şenay and A. Akan, "Asynchronous Representation and Processing of Nonstationary Signals : A Time-Frequency Framework", *IEEE Signal Processing Magazine*, vol. 30, pp. 42-52, Nov. 2013

[11] D. Slepian, "Prolate spheroidal wave functions, Fourier analysis, and uncertainty - V: The discrete case," *The Bell System Technical Journal*, vol. 57, no. 5, pp. 1371–1430, May/Jun. 1978.

[12] S. Şenay, J. Oh and L.F. Chaparro, "Regularized signal reconstruction for level-crossing sampling using Slepian functions", *Signal Processing*, pp. 1157-1165, Vol.92, 2012.

[13] E. Sejdić , A. Can, L.F. Chaparro, C.M Steele and T. Chau,"Compressive sampling of swallowing accelerometry signals using time-frequency dictionaries based on modulated discrete prolate spheroidal sequences", *EURASIP Journal on Advances in Signal Processing*, pp. 1-14, 2012.

[14] J.A. Hogan and J.D. Lakey, "Wavelet frames generated by bandpass prolate functions", *International Conference on Sampling Theory and Applications (SampTA)*, pp.120-123, May 2015.

[15] G.G. Walter and X.Shen, "Wavelets based on prolate spheroidal wave functions", *Journal of Fourier Analysis and Applications*, vol. 10, no.1, pp. 1-26, 2004.

[16] T. Zemen and C. F. Mecklenbräuker, "Time-variant channel estimation using discrete prolate spheroidal sequences," *IEEE Transactions on Signal Processing*, vol. 53, no. 9, pp. 3597–3607, Sep. 2005.

[17] E. Sejdić, M. Luccini, S. Primak, K. Baddour, and T. Willink, "Channel estimation using DPSS based frames," in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP 2008)*, Las Vegas, Nevada, USA, Mar./Apr. 31–4, 2008, pp. 2849–2852.

[18] G. Strang, T. Nguyen, "Wavelets and Filter Banks", *Wellesley-Cambridge Press*, Oct. 1996.

[19] M. Vetterli and C. Herley, "Wavelets and filter banks: theory and design", *IEEE Transaction on Signal Processing*, vol.40, no. 9, pp. 2207-2232, Sep. 1992.

[20] S.G. Mallat, "A theory of multiresolution signal decomposition: the wavelet representation", *IEEE Transaction on Pattern Analysis and Machine Intelligence*, vol. 11, no.7, pp. 674-693, Jul. 1989

[21] A. Cohen, I. Daubechies and J.C. Feauveau, "Biorthogonal bases of compactly supported wavelets", *Communications on pure and applied mathematics*, vol. 45, no.5, pp. 485-560, 1992.